Two Elementary Derivations of the Pure Fisher-Hartwig Determinant

Albrecht Böttcher and Harold Widom

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By the "pure Fisher-Hartwig determinant" we mean the Toeplitz determinant $D_n(\varphi) := \det(\varphi_{i-j})_{i,j=1}^n$ where the φ_k are the Fourier coefficients of

$$\varphi(z) = (1-z)^\alpha\,(1-z^{-1})^\beta,$$

a so-called pure Fisher-Hartwig singularity. The kth Fourier coefficient of φ equals

$$(-1)^k \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1-k)\,\Gamma(\beta+1+k)}. \tag{1}$$

The formula for the determinant is

$$D_n(\varphi) = G(n+1)\,\frac{G(\alpha+\beta+n+1)}{G(\alpha+\beta+1)}\,\frac{G(\alpha+1)}{G(\alpha+n+1)}\,\frac{G(\beta+1)}{G(\beta+n+1)}, \tag{2}$$

where G is the Barnes G-function. This was deduced by Silbermann and one of the authors [2] from a factorization of the Toeplitz matrix $T_n(\varphi)$ due to Duduchava and Roch. Another proof was recently found by Basor and Chen [1] using the theory of orthogonal polynomials, which motivated us to present the two proofs of this note.

First proof. This proof is analogous to the usual derivation of the Cauchy determinant and its philosophy is that the most elegant way to determine a rational function is to find its zeros and poles.

The factor $(-1)^k$ in (1) will not affect the determinant. We write the rest as

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\,\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+1-k)\,\Gamma(\beta+1+k)}.$$

For the evaluation of $D_n(\varphi)$ the first factor will contribute in the end the factor

$$\left(\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\right)^n. \tag{3}$$

The remaining factor gives the determinant of the matrix M with i,j entry

$$M_{ij} = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+1-i+j)\,\Gamma(\beta+1+i-j)}. \tag{4}$$

We think of $\det M$ as a function of α, with β as a parameter, and shall establish the following two facts:

(a) The only possible poles of $\det M$ (including ∞) are at $-1,\ldots,-n+1$, with the pole at $-k$ having order at most $n-k$.

(b) For $k=1,\ldots,n-1$ $\det M$ has a zero at $\alpha=-\beta-k$ of order at least $n-k$.

Granting these for the moment, let us derive (2). If $\det M$ had exactly the poles and zeros as stated it would be a constant depending on β times

$$\prod_{k=1}^{n-1} \frac{(\alpha+\beta+k)^{n-k}}{(\alpha+k)^{n-k}}.$$

If there were more zeros or fewer poles, then in the representation of $\det M$ as a quotient of polynomials there would be at least one more non-constant factor in the numerator than in the denominator. But then $\det M$ would not be analytic at $\alpha = \infty$, which we know it to be. Thus $\det M$ is a constant times the above. When $\alpha = 0$ the matrix is upper-triangular with diagonal entries all equal to 1, so $\det M = 1$ then. This determines the constant factor, and we deduce

$$\det M = \prod_{k=1}^{n-1} \frac{k^{n-k}\,(\alpha+\beta+k)^{n-k}}{(\alpha+k)^{n-k}(\beta+k)^{n-k}}.$$

Multiplying this by (3) gives (2). We now establish (a) and (b).

Proof of (a): The only possible finite poles of the M_{ij} arise from the poles of the numerator in (4) at the negative integers $-k$. The pole at $-k$ will not be cancelled by a pole in the denominator precisely when $j - i \geq k$. In particular for there to be a pole we must have $k \leq n-1$. The order of the pole at $\alpha = -k$ in a term $\prod M_{i,\sigma(i)}$ in the expansion of $\det M$ (here σ is a permutation of $0, \ldots, n-1$) equals

$$\#\{i : \sigma(i) \geq i + k\}.$$

Since the inequality can only occur when $i < n - k$ the above number is at most $n - k$. This establishes the statement about the possible finite poles. To see that $\det M$ is analytic at $\alpha = \infty$, we observe that the order of the pole of M_{ij} there equals $i - j$. (The order is counted as negative when there is a zero.) Hence the order of the pole there of $\prod M_{i,\sigma(i)}$ equals $\sum_i (i - \sigma(i)) = 0$.

Proof of (b): Let us write $M_{ij}(\alpha,\, \beta)$ instead of M_{ij} to show its dependence on α and β. A simple computation gives for $i = 1, \ldots, n-1$

$$\begin{aligned} &M_{i,j}(\alpha,\,\beta) + M_{i-1,j}(\alpha,\,\beta) \\ &= (\alpha+\beta+1)\,\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+2-i+j)\,\Gamma(\beta+1+i-j)} = \frac{\alpha+\beta+1}{\alpha+1}\,M_{i,j}(\alpha+1,\beta). \end{aligned}$$

In other words, if we add to each of the last $n-1$ rows of $M(\alpha,\,\beta)$ the preceding row we obtain $(\alpha+\beta+1)/(\alpha+1)$ times the last $n-1$ rows of the matrix $M(\alpha+1,\,\beta)$. Then we continue. If we apply these operations a total of k times the last $n-k$ rows of the matrix obtained from $M(\alpha,\,\beta)$ in this way (which does not change its rank) is equal to

$$\frac{(\alpha+\beta+1)\cdots(\alpha+\beta+k)}{(\alpha+1)\cdots(\alpha+k)}$$

times the last $n-k$ rows of the matrix $M(\alpha+k,\,\beta)$. It follows that if we set $\alpha = -\beta-k$ in $M(\alpha,\,\beta)$ we get a matrix of rank at most k. From this it follows that if we differentiate

$\det M(\alpha,\,\beta)$ up to $n-k-1$ times with respect to α and set $\alpha=-\beta-k$ we get zero. Thus there is a zero there of order at least $n-k$.

Second proof. This proof does not aspire to elegance but is rather the simple endeavor to go ahead straightforwardly.

Taking into account formula (1) for the Fourier coefficients of φ we get

$$D_n(\varphi)=(\Gamma(\alpha+\beta+1))^n \det\left(\frac{1}{\Gamma(\alpha+1-i+j)\Gamma(\beta+1+i-j)}\right)_{i,j=1}^n.$$

Extracting the factor $1/\Gamma(\alpha+1+n-i)$ from the ith row and $1/\Gamma(\beta+1+n-j)$ from the jth column, we obtain

$$\begin{aligned}\frac{D_n(\varphi)}{(\Gamma(\alpha+\beta+1))^n}&=\prod_{i=1}^n\frac{1}{\Gamma(\alpha+1+n-i)}\,\prod_{j=1}^n\frac{1}{\Gamma(\beta+1+n-j)}\,D_n(\alpha,\beta)\\&=\frac{G(\alpha+1)}{G(\alpha+n+1)}\,\frac{G(\beta+1)}{G(\beta+n+1)}\,D_n(\alpha,\beta)\end{aligned}\tag{5}$$

with

$$D_n(\alpha,\beta)=\det\left(\prod_{\ell=1}^{n-j}(\alpha-i+j+\ell)\,\prod_{k=1}^{n-i}(\beta+i-j+k)\right)_{i,j=1}^n.$$

The last row of $D_n(\alpha,\beta)$ is

$$\left(\begin{array}{cccccc}\prod_{\ell=0}^{n-2}(\alpha-\ell) & \prod_{\ell=0}^{n-3}(\alpha-\ell) & \dots & (\alpha-1)\alpha & \alpha & 1\end{array}\right).$$

With the objective that the last row becomes $(\,0\;0\;\dots\;0\;1\,)$, we subtract $\alpha-n+2$ times column 2 from column 1, $\alpha-n+3$ times column 3 from column 2, $\dots$, and finally α times column n from column $n-1$. What results is that

$$D_n(\alpha,\beta)=(n-1)!\,(\alpha+\beta+1)^{n-1}\,D_{n-1}(\alpha+1,\beta).$$

Since $D_1(\alpha+n-1,\beta)=1$, it follows that

$$\begin{aligned}D_n(\alpha,\beta)&=\prod_{k=1}^{n-1}(n-k)!\,(\alpha+\beta+k)^{n-k}=\prod_{\ell=1}^n\Gamma(\ell)\,\prod_{\ell=1}^n\frac{\Gamma(\alpha+\beta+\ell)}{\Gamma(\alpha+\beta+1)}\\&=G(n+1)\,\frac{G(\alpha+\beta+n+1)}{G(\alpha+\beta+1)}\,\frac{1}{(\Gamma(\alpha+\beta+1))^n}.\end{aligned}\tag{6}$$

Inserting (6) in (5) we arrive at the desired formula.

References

[1] E. L. Basor and Y. Chen, *Toeplitz determinant from compatibility conditions*, preprint.

[2] A. Böttcher and B. Silbermann, *Toeplitz matrices and determinants with Fisher-Hartwig symbols*, J. Funct. Anal. **62** (1985), 178–214.

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Fakultät für Mathematik
TU Chemnitz
09107 Chemnitz, Germany
e-mail: aboettch@mathematik.tu-chemnitz.de

Department of Mathematics
University of California
Santa Cruz, CA 95064, USA
e-mail: widom@math.ucsc.edu